Journal of Mathematical Analysis and Applications **246**, 409–422 (2000)
doi:10.1006/jmaa.2000.6795, available online at http://www.idealibrary.com on IDEAL®

Generalized Set-Valued Variational Inclusions in Banach Spaces[1]

S. S. Chang

Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, People's Republic of China

Y. J. Cho

Department of Mathematics, Gyeongsang National University, Chinju 660-701, Korea

B. S. Lee

Department of Mathematics, Kyungsung University, Pusan 608-736, Korea

and

I. H. Jung

Department of Mathematics, Pusan National University, Pusan 609-735, Korea

Submitted by Muhammad Aslam Noor

Received March 19, 1999

In this paper, a generalization and improvement of Noor's theorem for generalized set-valued variational inclusion in real Banach spaces is studied. The results presented in this paper generalize, improve, and unify a number of recent results.

Key Words: variational inclusion; variational inequality; iterative method; resolvent equation; m-accretive mapping; accretive mapping.

[1] This work was done while the first author visited Pusan National University, Korea, and was supported by the Korea Science and Engineering Foundation, National Natural Science Foundation of China. The second and third authors were supported in part by the Korea Research Foundation made in the program year of 1998, Project 1998-015-D00020.

0022-247X/00 $35.00

1. INTRODUCTION

In recent years, variational inequalities have been extended and generalized in different directions, using novel and innovative techniques. Useful and important generalizations of variational inequalities are variational inclusions, which have been studied by Huang [9], Moudafi and Noor [10], Noor [15] and Uko [24] in the Hilbert spaces settings.

Recently, in [15], Noor introduced and studied the following class of important generalized set-valued variational inclusion problems in a Hilbert space H:

For a given maximal monotone mapping A: $H \to H$, a nonlinear mapping $N(\cdot, \cdot)$: $H \times H \to H$, set-valued mappings T, V: $H \to C(H)$, and a single-valued mapping g: $H \to H$, find $u \in H$, $w \in T(u)$, $y \in V(u)$ such that

$$\theta \in N(w, y) + A(g(u)), \tag{1.1}$$

where $C(H)$ denotes the family of all nonempty compact subsets of H.

Inspired and motivated by the results in Noor [15, 21], the purpose of this paper is to introduce and study a class of more general set-valued variational inclusions without the compactness condition in Banach spaces. By using the general duality principle for the sum of two operators given in Noor [15], we also establish the equivalence between the generalized set-valued variational inclusions and the resolvent equations in Banach spaces. We use the equivalence technique of Noor [15], Nadler's theorem [11], and our inequality [1, 2] to suggest an iterative method for solving the generalized set-valued variational inclusions in real Banach spaces. The results presented in this paper generalize, improve, and unify the corresponding results of Chang et al. [3, 5], Hassouni and Moudafi [7], Huang [8, 9], Noor [14–18, 21], and Zeng [25].

2. PRELIMINARIES

Throughout this paper, we assume that E is a real Banach space, E^* is the topological dual space of E, $CB(E)$ is the family of all nonempty closed and bounded subsets of E, $D(\cdot, \cdot)$ is the Hausdorff metric on $CB(E)$ defined by

$$D(A, B) = \max\left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y) \right\},$$

$\langle \cdot, \cdot \rangle$ is the dual pair between E and E^*, $D(T)$ denotes the domain of T, and J: $E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{ f \in E^* : \langle x, f \rangle = \|x\| \cdot \|f\|, \|f\| = \|x\| \}, \qquad x \in E.$$

DEFINITION 2.1 [1]. Let A: $D(A) \subset E \to 2^E$ be a set-valued mapping.

(1) The mapping A is said to be accretive if, for any $x, y \in D(A)$, $u \in Tx$, $v \in Ty$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle u - v, j(x-y) \rangle \geq 0.$$

(2) The mapping A is said to be k-strongly accretive, $k \in (0,1)$, if, for any $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that, for any $u \in Tx$, $v \in Ty$,

$$\langle u - v, j(x-y) \rangle \geq k\|x-y\|^2.$$

(3) The mapping A is said to be m-accretive if A is accretive and $(I + \varrho A)(D(A)) = E$ for every (equivalently, for some) $\varrho > 0$, where I is the identity mapping (equivalently, if A is accretive and $(I + A)(D(A)) = E$).

Remark 2.1. It is well known that, if $E = E^* = H$ is a Hilbert space, then the notion of an accretive mapping coincides with that of a monotone mapping [1].

Thus we have the following:

PROPOSITION 2.1. *Let $E = H$ be a Hilbert space. Then A: $D(A) \subset H \to 2^H$ is an m-accretive mapping if and only if A: $D(A) \subset H \to 2^H$ is a maximal monotone mapping.*

Proof. ($\Rightarrow$) Let A: $D(A) \subset H \to 2^H$ be an m-accretive mapping. Then we prove that A is a maximal monotone mapping. Suppose that A is not maximal monotone. Then there exists $(x_0, u_0) \in H \times H$ such that

(1) $u_0 \in Ax_0$,

(2) $\langle u_0 - u, x_0 - x \rangle \geq 0$ for all $x \in D(A)$ and $u \in Ax$.

Since $(I + A)(D(A)) = H$, there exist $x_1 \in D(A)$ and $u_1 \in Ax_1$ such that

(3) $x_1 + u_1 = x_0 + u_0$, i.e., $u_0 - u = x_1 - x_0$.

Taking $u = u_1$, $x = x_1$ in (2) and using (3), we have

$$\begin{aligned} 0 &\leq \langle u_0 - u_1, x_0 - x_1 \rangle \\ &= \langle x_1 - x_0, x_0 - x_1 \rangle \\ &= -\|x_1 - x_0\|^2 \\ &\leq 0, \end{aligned}$$

which implies that $x_1 = x_0$ and $u_1 = u_0$. Hence we have $x_0 \in D(A)$ and $u_0 \in Ax_0$, which contradicts (1).

($\Leftarrow$) If $A: D(A) \subset H \to 2^H$ is maximal monotone, by [1, Theorem 1.2], we have $(I + A)(D(A)) = H$. This implies that A is m-accretive. ▮

PROBLEM 2.1. Let E be a real Banach space. Let $T, V: E \to CB(E)$ be two set-valued mappings and $g: E \to E$ be a single-valued mapping. For a given m-accretive mapping $A: E \to 2^E$ and a nonlinear mapping $N(\cdot, \cdot): E \times E \to E$, we consider the following problem:

Find $u \in E$, $w \in T(u)$, $y \in V(u)$ such that

$$\theta \in N(w, y) + A(g(u)). \tag{2.1}$$

The problem (2.1) is called the generalized set-valued variational inclusion problem in Banach spaces.

Now we consider some special cases of the problem (2.1):

(1) If $E = H$ is a Hilbert space and $A: H \to H$ is a maximal monotone mapping, then by Proposition 2.1, A is an m-accretive mapping. Thus the problem (2.1) is equivalent to finding $u \in H$, $w \in T(u)$, $y \in V(u)$ such that

$$\theta \in N(w, y) + A(g(u)). \tag{2.2}$$

This problem is called the generalized set-valued variational inclusion, which was introduced and studied in Noor [15] by using the compactness condition and the resolvent equation technique.

(2) If $g \equiv I$, the identity mapping, then the problem (2.1) is equivalent to finding $u \in E$, $w \in Tu$, $y \in V(u)$ such that

$$\theta \in N(w, y) + A(u), \tag{2.3}$$

which is called the set-valued variational inclusion problem in Banach spaces. In the setting of Hilbert spaces, this problem has been studied by Noor [15].

(3) If $E = H$ is a Hilbert space and $A = \partial\varphi$, the subdifferential of a proper convex lower semicontinuous functional $\varphi: H \to \mathbb{R} \cup \{+\infty\}$, then the problem (2.2) is equivalent to finding $u \in H$, $w \in Tu$, $y \in V(u)$ such that

$$\langle N(w, y), v - g(u) \rangle \geq \varphi(g(u)) - \varphi(v) \tag{2.4}$$

for all $v \in H$. This problem is called the generalized set-valued mixed variational inequality, which was introduced and studied by Noor et al. [21].

We remark that, if the proper, convex, and lower semicontinuous φ is the indicator function of a closed convex set K in the Hilbert space H, then the problem (2.1) is equivalent to finding $u \in H$, $g(u) \in K$, $w \in T(u)$, and $y \in V(u)$ such that

$$\langle N(w, y), v - g(u) \rangle \geq 0 \tag{2.5}$$

for all $v \in K$, which is called the generalized set-valued variational inequality. This problem has been studied by Noor [18] using the Wiener–Hopf equation technique.

Recently, this problem with N being some special case was also considered in the setting of Banach spaces [3].

Summing up the above arguments, it shows that, for a suitable choice of the mapping T, V, A, g, N, φ and the space E, we can obtain a number of known and new classes of variational inequalities, variational inclusions, and the corresponding optimization problems from the generalized set-valued variational inclusion (2.1).

For the sake of convenience, next we recall some definitions and notions.

DEFINITION 2.2 [1]. Let A: $D(A) \subset E \to 2^E$ be an m-accretive mapping. For any $\rho > 0$, the mapping R_A: $E \to D(A)$ associated with A defined by

$$R_A(u) = (I + \varrho A)^{-1}(u), \qquad u \in D(A), \tag{2.6}$$

is called the resolvent operator.

Remark 2.2. It is well known that R_A is a single-valued and nonexpansive mapping (see Barbu [1]).

DEFINITION 2.3. Let T, V: $E \to 2^E$ be two set-valued mappings and $N(\cdot, \cdot)$: $E \times E \to E$ be a nonlinear mapping.

(1) The mapping $x \mapsto N(x, y)$ is said to be β-Lipschitzian continuous with respect to the mapping T if, for any $x_1, x_2 \in E$ and $w_1 \in Tx_1$, $w_2 \in Tx_2$,

$$\|N(w_1, y) - N(w_2, y)\| \leq \beta \|x_1 - x_2\|, \qquad y \in E,$$

where $\beta > 0$ is a constant.

(2) The mapping $y \mapsto N(x, y)$ is said to be γ-Lipschitzian continuous with respect to the mapping V if, for any $u_1, u_2 \in E$ and $v_1 \in V(u_1)$, $v_2 \in V(u_2)$,

$$\|N(x, v_1) - N(x, v_2)\| \leq \gamma \|u_1 - u_2\|, \qquad x \in E,$$

where $\gamma > 0$ is a constant.

DEFINITION 2.4. Let A: $E \to CB(E)$ be a set-valued mapping and $D(\cdot, \cdot)$ be the Hausdorff metric on $CB(E)$. T is said to be ξ-Lipschitzian continuous if, for any $x, y \in E$,

$$D(Tx, Ty) \leq \xi \|x - y\|,$$

where $\xi > 0$ is a constant.

Related to the generalized set-valued variational inclusion (2.1) in Banach spaces, we consider the following problem:

Find $z, u \in E$, $w \in Tu$, $y \in V(u)$ such that

$$N(w, y) + \varrho^{-1} F_A z = 0, \tag{2.7}$$

where $\varrho > 0$ is a constant and $F_A = (I - R_A)$, where I is the identity operator and R_A is the resolvent operator. The equation of the type (2.7) is called the resolvent equation in Banach spaces. If $E = H$ is a Hilbert space, then the problem (2.7) was introduced and studied by Noor [15]. It has been shown in [15] that the problems (2.2) and (2.7) are equivalent. It is worth mentioning that the theory of resolvent equations is mainly due to Noor and, for more information, see Noor [13] and references therein. It has been shown that the resolvent is equivalent to the mixed variational inequalities. This equivalence has played a significant and fundamental part in suggesting various iterative methods for solving variational inequalities and variational inclusions. For recent applications, sensitivity analysis, and numerical methods, see [10, 13, 19, 20] and references therein. In particular, if $E = H$ is a Hilbert space and $A = \partial\varphi$, where φ is the indicator function for a closed subset K of H, then the resolvent operator $R_A = P_K$, the projection of H onto K. Therefore, the problem (2.7) is equivalent to finding $z, u \in H$, $w \in T(u)$, $y \in V(u)$ such that

$$N(w, y) + \rho^{-1} Q_K z = 0, \tag{2.8}$$

where $Q_K = (I - P_K)$ and $\rho > 0$ is a constant. The equation (2.8) is called the generalized Wiener–Hopf equation, which was introduced and studied by Noor [18].

It has been shown in [18] that the problems are equivalent using the project technique. Furthermore, it is clear that the resolvent equations are more general than the Wiener–Hopf equations and include the Wiener–Hopf equations as a special case. For the applications, formulation, sensitivity analysis, motivation, and numerical methods for the Wiener–Hopf equations, see [12, 13, 22, 23].

The following two lemmas play an important role in proving our main results.

LEMMA 2.1 [1, 2]. *Let E be a real Banach space and $J: E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $x, y \in E$,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle$$

for all $j(x + y) \in J(x + y)$.

Proof. It is well known that $J(x) = \partial\psi(x)$, where $\psi(x) = \frac{1}{2}\|x\|^2$ for all $x \in E$. By the definition of the subdifferential of ψ, for every $x, y \in E$ and $j(x + y) \in J(x + y)$, we have

$$\psi(x) - \psi(x + y) \geq \langle x - (x + y), j(x + y)\rangle,$$

i.e.,

$$\|x\|^2 - \|x + y\|^2 \geq -2\langle y, j(x + y)\rangle.$$

Therefore, it follows that

$$\|x + y\| \leq \|x\|^2 + 2\langle y, j(x + y)\rangle$$

for all $j(x + y) \in J(x + y)$. This completes the proof. ∎

LEMMA 2.2. *The following conclusions are equivalent:*

(i) (u, w, y), *where $u \in E$, $w \in T(u)$, and $y \in V(u)$, is a solution of the set-valued variational inclusion* (2.1),

(ii) (u, w, y) *is a solution of the equation*

$$g(u) = R_A(g(u) - \rho N(w, y)), \tag{2.9}$$

(iii) (z, u, w, y) *is a solution of the resolvent equation* (2.7), *where*

$$\begin{aligned} z &= g(u) - \rho N(w, y), \\ g(u) &= R_A z. \end{aligned} \tag{2.10}$$

Proof. If we use the technique given in Noor [15], we can prove this lemma immediately.

We now invoke Lemma 2.2 and (2.10) to suggest the following algorithms for solving the generalized set-valued variational inclusion (2.1) in Banach spaces.

ALGORITHM 2.1. *For any given z_0, $u_0 \in E$, $w_0 \in T(u_0)$, $y_0 \in V(u_0)$, from* (2.10), *let*

$$z_1 = g(u_0) - \rho N(w_0, y_0).$$

Take $u_1 \in E$ such that

$$g(u_1) = R_A z_1.$$

Since $w_0 \in T(u_0)$ *and* $y_0 \in V(u_0)$, *by Nadler's theorem* [11], *there exist* $w_1 \in Tu_1$ *and* $y_1 \in V(u_1)$ *such that*

$$\|w_0 - w_1\| \le (1 + 1) D(T(u_0), T(u_1)),$$
$$\|y_0 - y_1\| \le (1 + 1) D(V(u_0), V(u_1)),$$

where D *is the Hausdorff metric on* $CB(E)$. *Let*

$$z_2 = g(u_1) - \rho N(w_1, y_1)$$

and take $u_2 \in E$ *such that*

$$g(u_2) = R_A(z_2).$$

Again by Nadler's theorem, there exist $w_2 \in T(u_2)$ *and* $y_2 \in V(u_2)$ *such that*

$$\|w_1 - w_2\| \le \left(1 + \tfrac{1}{2}\right) D(T(u_1), T(u_2)),$$
$$\|y_1 - y_2\| \le \left(1 + \tfrac{1}{2}\right) D(V(u_1), V(u_2)).$$

Continuing in this way, we can obtain the following:

For any given z_0, $u_0 \in E$, $w_0 \in T(u_0)$, $y_0 \in V(u_0)$, *compute the sequences* $\{z_n\}$, $\{u_n\}$, $\{w_n\}$, *and* $\{y_n\}$ *by iterative schemes such that*

$$\begin{aligned}
&\text{(i)} \quad g(u_n) = R_A z_n, \\
&\text{(ii)} \quad w_n \in T(U_n), \qquad \|w_n - w_{n+1}\| \\
&\qquad \le \left(1 + \frac{1}{n+1}\right) D(T(u_n), T(u_{n+1})), \\
&\text{(iii)} \quad y_n \in V(u_n), \qquad \|y_n - y_{n+1}\| \\
&\qquad \le \left(1 + \frac{1}{n+1}\right) D(V(u_n), T(u_{n+1})), \\
&\text{(iv)} \quad z_{n+1} = g(u_n) - \rho N(w_n, y_n), \qquad n = 0, 1, 2, \dots .
\end{aligned} \tag{2.11}$$

Remark 2.3. It should be pointed out that, if $E = H$ is a Hilbert space and $A = \partial\phi$, where ϕ is the indicator function of a closed convex subset K of H, then $R_A = P_K$, the projection of H onto the closed convex set K in H. Consequently, Algorithm 2.1 reduces to the following method for solving the set-valued variational inequality (2.5).

ALGORITHM 2.2. *For any given z_0, $u_0 \in E$, $w_0 \in T(u_0)$, and $y_0 \in V(u_0)$, compute the sequences $\{z_n\}$, $\{u_n\}$, $\{w_n\}$, and $\{y_n\}$ by the iterative schemes such that*

$$\begin{aligned}
&g(u_n) = P_K z_n, \\
&w_n \in T(u_n), \qquad \|w_n - w_{n+1}\| \leq \left(1 + \frac{1}{n+1}\right) D(T(u_n), T(u_{n+1})), \\
&y_n \in V(u_n), \qquad \|y_n - y_{n+1}\| \leq \left(1 + \frac{1}{n+1}\right) D(V(u_n), T(u_{n+1})), \\
&z_{n+1} = g(u_n) - \rho N(w_n, y_n), \qquad n = 0, 1, 2, \ldots .
\end{aligned} \tag{2.12}$$

3. MAIN RESULTS

In this section, we study the convergence analysis of Algorithm 2.1.

THEOREM 3.1. *Let E be a real Banach space, $T, V\colon E \to CB(E)$, $A\colon E \to 2^E$ be three set-valued mappings, $N(\cdot,\cdot)\colon E \times E \to E$ be a single-valued continuous mapping, and $g\colon E \to E$ be a single-valued mapping satisfying the following conditions*:

(i) *g is σ-Lipschitzian continuous and $(g - I)$ is k-strongly accretive, where $\sigma > 0$ and $k \in (0,1)$ both are constants,*

(ii) *$A\colon E \to 2^E$ is m-accretive,*

(iii) *$T\colon E \to CB(E)$ is μ-Lipschitzian continuous,*

(iv) *$V\colon E \to CB(E)$ is ξ-Lipschitzian continuous,*

(v) *the mapping $x \mapsto N(x, y)$ is β-Lipschitzian continuous with respect to the set-valued mapping T for any given $y \in E$,*

(vi) *the mapping $y \mapsto N(x, y)$ is γ-Lipschitzian continuous with respect to the set-valued mapping V for any given $x \in E$, where all μ, ξ, β, γ are positive constants.*

If the following conditions are satisfied,

$$\begin{aligned}
&1 \leq \sigma < \sqrt{1 + 2k}, \qquad 0 < \xi, \qquad \mu < \frac{1}{2}, \\
&0 < \rho < \min\left\{\frac{1}{\beta + \gamma}, \frac{(2k+1) - \sigma^2}{2(k+1)(\beta + \gamma)}\right\},
\end{aligned} \tag{3.1}$$

then there exist $z, u \in E$, $w \in T(u)$, $y \in V(u)$ *satisfying the resolvent equation* (2.7), *and the iterative sequences* $\{z_n\}$, $\{u_n\}$, $\{w_n\}$, *and* $\{y_n\}$ *generated by Algorithm* 2.1 *converge strongly to* z, u, w, y *in* E, *respectively*.

Proof. From (iv) of (2.11), the conditions (i), (v), (iv), and Lemma 2.1, it follows that, for any $j(z_{n+1} - z_n) \in J(z_{n+1} - z_n)$,

$$\begin{aligned}
&\|z_{n+1} - z_n\|^2 \\
&\quad = \left\| g(u_n) - g(u_{n-1}) - \rho\{N(w_n, y_n) - N(w_{n-1}, y_{n-1})\} \right\|^2 \\
&\quad \le \|g(u_n) - g(u_{n-1})\|^2 \\
&\qquad - 2\rho\langle N(w_n, y_n) - N(w_{n-1}, y_{n-1}), j(z_{n+1} - z_n)\rangle \\
&\quad \le \sigma^2\|u_n - u_{n-1}\|^2 \\
&\qquad + 2\rho\| N(w_n, y_n) - N(w_{n-1}, y_n) + N(w_{n-1}, y_n) - N(w_{n-1}, y_{n-1}) \| \\
&\qquad \cdot \|z_{n+1} + z_n\| \\
&\quad \le \sigma^2\|u_n - u_{n-1}\|^2 + 2\rho\{ \beta\|u_n - u_{n-1}\| + \gamma\|u_n - u_{n-1}\|\}\|z_{n+1} - z_n\| \\
&\quad \le \sigma^2\|u_n - u_{n-1}\|^2 + \rho(\beta + \gamma)\left\{\|u_n - u_{n-1}\|^2 + \|z_{n+1} - z_n\|^2\right\},
\end{aligned}$$

which implies that

$$\|z_{n+1} - z_n\|^2 \le \frac{\sigma^2 + \rho(\beta + \gamma)}{1 - \rho(\beta + \gamma)}\|u_n - u_{n-1}\|^2. \tag{3.2}$$

Also from (i) of (2.11), the condition (i), and Lemma 2.1, it follows that, for any $j(u_n - u_{n-1}) \in J(u_n - u_{n-1})$,

$$\begin{aligned}
&\|u_n - u_{n-1}\|^2 \\
&\quad = \left\|(R_A z_n - R_A z_{n-1}) - \left[g(u_n) - u_n - (g(u_{n-1}) - u_{n-1})\right]\right\|^2 \\
&\quad \le \|R_A z_n - R_A z_{n-1}\|^2 \\
&\qquad - 2\langle g(u_n) - u_n - (g(u_{n-1}) - u_{-1}), j(u_n - u_{n-1})\rangle \\
&\quad \le \|z_n - z_{n-1}\|^2 - 2k\|u_n - u_{n-1}\|^2,
\end{aligned}$$

which implies that

$$\|u_n - u_{n-1}\|^2 \le \frac{1}{1 + 2k}\|z_n - z_{n-1}\|^2. \tag{3.3}$$

Substituting (3.3) into (3.2), we have

$$\|z_{n+1} - z_n\|^2 \le \frac{\sigma^2 + \rho(\beta + \gamma)}{(1 + 2k)(1 - \rho(\beta + \gamma))}\|z_n - z_{n-1}\|^2,$$

i.e.,

$$\|z_{n+1} - z_n\| \le \alpha \|z_n - z_{n-1}\|, \tag{3.4}$$

where

$$\alpha = \left\{ \frac{\sigma^2 + \rho(\beta + \gamma)}{(1 + 2k)(1 - \rho(\beta + \gamma))} \right\}^{1/2}.$$

Now we prove that $0 < \alpha < 1$. In fact, from the condition (3.1), it follows that

$$0 < \rho < \frac{1}{\beta + \gamma}, \qquad 0 < 2k + 1 - \sigma^2,$$

and

$$2(k + 1)(\beta + \gamma)\rho < (2k + 1) - \sigma^2,$$

which implies that

$$\sigma^2 + 2\rho(\beta + \gamma) - 1 < 2k[1 - \rho(\beta + \gamma)].$$

This implies that

$$0 < \frac{\sigma^2 + 2\rho(\beta + \gamma) - 1}{2[1 - \rho(\beta + \gamma)]} < k < 1,$$

and so $0 < \alpha < 1$.

Therefore, $\{z_n\}$ is a Cauchy sequence in E. Since E is a Banach space, there exists $z \in E$ such that $z_n \to z$ as $n \to \infty$. From (3.3), we know that the sequence $\{u_n\}$ is also a Cauchy sequence in E. Therefore, there exists $u \in E$ such that $u_n \to u$ as $n \to \infty$.

On the other hand, it follows from (ii), (iii) of (2.11) and the conditions (iii) and (iv) that

$$\begin{aligned}\|w_n - w_{n+1}\| &\le \left(1 + \frac{1}{n+1}\right) D(T(u_n), T(u_{n+1}))\\ &\le \left(1 + \frac{1}{n+1}\right) \mu \|u_n - u_{n+1}\|,\\ \|y_n - y_{n+1}\| &\le \left(1 + \frac{1}{n+1}\right) D(V(u_n), V(u_{n+1}))\\ &\le \left(1 + \frac{1}{n+1}\right) \xi \|u_n - u_{n+1}\|.\end{aligned}$$

By the condition (3.1), these inequalities imply that the sequences $\{w_n\}$ and $\{y_n\}$ both are Cauchy sequences in E. Hence there exist w and $y \in E$ such that $w_n \to w$ and $y_n \to y$ as $n \to \infty$, respectively. By the continuity of g and N, it follows from (iv) of (2.11) that

$$z_{n+1} = g(u_n) - \rho N(w_n, y_n) \to z = g(u) - \rho N(w, y) \qquad (n \to \infty), \tag{3.5}$$

$$R_A z_n = g(u_n) \to g(u) = R_A(z) \qquad (n \to \infty). \tag{3.6}$$

By (3.5), (3.6), and Lemma 2.2(iii), we have

$$N(w, y) + \rho^{-1}(I - R_A)z = 0.$$

Finally, we prove that $w \in T(u)$ and $y \in V(u)$. In fact, since $w_n \in T(u_n)$ and

$$\begin{aligned} d(w_n, T(u)) &\le \max\left\{d(w_n, T(u)), \sup_{x \in T(u)} d(T(u_n), x)\right\} \\ &\le \max\left\{\sup_{y \in T(u_n)} d(y, T(w)), \sup_{x \in T(u)} d(T(u_n), x)\right\} \\ &= D(T(u_n), T(u)), \end{aligned}$$

we have

$$\begin{aligned} d(w, T(u)) &\le \|w - w_n\| + d(w_n, T(u)) \\ &\le \|w - w_n\| + D(T(u_n), T(u)) \\ &\le \|w - w_n\| + \mu\|u_n - u\| \to 0 \qquad (n \to \infty), \end{aligned}$$

which implies that $d(w, T(u)) = 0$ and so, since $T(u) \in CB(E)$, it follows that $w \in T(u)$. In a similar way, we can also prove that $y \in V(u)$.

By Lemma 2.2, it follows that (u, w, y) is a solution of the generalized set-valued variational inclusion problem (2.1) in real Banach spaces. This completes the proof. ∎

Remark 3.1. Theorem 3.1 extends and improves a number of the corresponding results in Huang [8, Theorems 4.1, 4.2], Noor [15, Theorem 4.2], Noor, Noor, and Rassias [21, Theorem 4.1], and the corresponding results of Chang et al. [3, 5], Huang [9], Noor [14, 16, 18], Zeng [25] from Hilbert spaces to real Banach spaces, and also removes the compactness condition in the main results of Noor [15, 21]. Further, the proof methods given in this paper are quite different from the methods given in those papers [15, 21].

The following result can be obtained from Theorem 3.1 immediately:

THEOREM 3.2. *Let H be a real Hilbert space. Suppose the following conditions are satisfied*:

(i) *$g: H \to H$ is a σ-Lipschitzian continuous mapping and, for any given $x, y \in H$,*

$$((g - I)(x) - (g - I)(y), x - y) \geq k\|x - y\|^2,$$

where $k \in (0, 1)$ is a constant,

(ii) *$A = \partial\phi: H \to 2^H$ is a maximal monotone operator, where $\phi: H \to \mathbb{R} \cup \{+\infty\}$ is a proper convex lower semicontinuous functional,*

(iii) *$T: H \to CB(H)$ is a μ-Lipschitzian continuous mapping,*

(iv) *$V: H \to CB(H)$ is a ξ-Lipschitzian continuous mapping,*

(v) *$N(\cdot, \cdot): H \times H \to H$ is a continuous mapping and the mapping $x \mapsto N(x, y)$ is β-Lipschitzian continuous with respect to the mapping T,*

(vi) *the mapping $y \mapsto N(x, y)$ is γ-Lipschitzian continuous with respect to the mapping V, where σ, μ, ξ, β, γ all are positive constants.*

If the following conditions are satisfied:

$$\begin{gathered} 1 \leq \sigma < \sqrt{1 + 2k}\,, \qquad 0 < \xi, \qquad \mu < \frac{1}{2}, \\ 0 < \rho < \min\left\{\frac{1}{\beta + \gamma}, \frac{(2k + 1) - \sigma^2}{2(k + 1)(\beta + \gamma)}\right\}, \end{gathered} \tag{3.7}$$

then there exist $u \in H$, $w \in T(u)$, $y \in V(u)$ satisfying the generalized set-valued mixed variational inequality (2.4), *and the iterative sequences* $\{u_n\}$, $\{w_n\}$, *and* $\{y_n\}$ *generated by Algorithm* 2.1 *converge strongly to u, w, y in H, respectively.*

Remark 3.2. Theorem 3.2 also extends and improves the corresponding results in Noor et al. [21] and Noor [18].

ACKNOWLEDGMENT

The authors express their thanks to the referee for his helpful suggestions.

REFERENCES

1. V. Barbu, "Nonlinear Semigroups and Differential Equations in Banach Spaces," Noordhoff Internat. Publ., Leyden, The Netherlands, 1976.
2. S. S. Chang, Some problems and results in the study of nonlinear analysis, *Nonlinear Anal. TMA* **30** (1997), 4197–4208.

3. S. S. Chang, The Mann and Ishikawa iterative approximation of solutions to variational inclusions with accretive type mappings, *Comput. Math. Appl.* **37**(9) (1999), 17–24.
4. S. S. Chang, Y. J. Cho, J. S. Jung, and S. M. Kang, Iterative approximations of fixed points and solutions for strongly accretive and strongly pseudo-contractive mappings in Banach spaces, *J. Math. Anal. Appl.* **224** (1998), 149–165.
5. S. S. Chang, X. Z. Yuan, F. Wang, and X. Long, The study of algorithms and convergence for generalized multi-valued quasi-variational inclusions, *Topological Methods in Nonlinear Anal.*, to appear.
6. K. Deimling, "Nonlinear Functional Analysis," Springer-Verlag, Berlin, 1985.
7. A. Hassouni and A. Moudafi, A perturbed algorithm for variational inclusions, *J. Math. Anal. Appl.* **185** (1994), 706–712.
8. N. J. Huang, On the generalized implicit quasi-variational inequalities, *J. Math. Anal. Appl.* **216** (1997), 197–210.
9. N. J. Huang, Generalized nonlinear variational inclusions with noncompact valued mappings, *Appl. Math. Lett.* **9**(3) (1996), 25–29.
10. A. Moudafi and M. A. Noor, Sensitivity analysis for variational inclusions by Wiener–Hopf equation technique, *J. Appl. Math. Stochastic Anal.* **12** (1999), 223–232.
11. S. B. Nadler, Multi-valued contraction mappings, *Pacific J. Math.* **30** (1969), 475–488.
12. M. A. Noor, Some recent advances in variational inequalities, Part I, basic concepts, *New Zealand J. Math.* **26** (1997), 53–80.
13. M. A. Noor, Some recent advances in variational inequalities, Part II, other concepts, *New Zealand J. Math.* **26** (1997), 229–255.
14. M. A. Noor, Sensitivity analysis of quasi-variational inequalities, *J. Optim. Theory Appl.* **95**(2) (1997), 399–407.
15. M. A. Noor, Generalized set-valued variational inclusions and resolvent equations, *J. Math. Anal. Appl.* **228** (1998), 206–220.
16. M. A. Noor, Numerical methods for monotone mixed variational inequalities, *Adv. Nonlinear Variat. Inequal.* **1** (1998), 51–79.
17. M. A. Noor, An implicit method for mixed variational inequalities, *Appl. Math. Lett.* **11**(4) (1998), 109–113.
18. M. A. Noor, Generalized multi-valued quasi-variational inequalities (II), *Comput. Math. Appl.* **35**(5) (1998), 63–78.
19. M. A. Noor, Some algorithms for general monotone mixed variational inequalities, *Math. Computer Modelling* **29**(7) (1999), 1–9.
20. M. A. Noor and K. I. Noor, Sensitivity analysis for quasi-variational inclusions, *J. Math. Anal. Appl.* **236** (1999), 290–299.
21. M. A. Noor, K. I. Noor, and Th. M. Rassias, Set-valued resolvent equations and mixed variational inequalities, *J. Math. Anal. Appl.* **220** (1998), 741–759.
22. S. M. Robinson, Normal maps induced by linear transformations, *Math. Oper. Res.* **17** (1992), 691–714.
23. P. Shi, Equivalence of variational inequalities with Wiener–Hopf equations, *Proc. Amer. Math. Soc.* **111** (1991), 339–346.
24. L. U. Uko, Strongly nonlinear generalized equations, *J. Math. Anal. Appl.* **220** (1998), 65–76.
25. L. U. Zeng, Iterative algorithm for finding approximate solutions to completely generalized strongly nonlinear quasi-variational inequality, *J. Math. Anal. Appl.* **201** (1996), 180–191.